PROCEEDINGS ITB Vol. 8, No. 3, 1974.

SCHOUTEN BRACKET OF HOLOMORPHIC TENSORS OF A KAHLERIAN MANIFOLD*)

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R I N G K A S A N

Kita perlihatkan bahwa "Schouten bracket" dari pada tensor-tensor holomorph milik suatu manifold kähler yang kompak W *mendefinisikan suatu struktur "graded complex Lie algebra" pada ruang tensor - tensor holomorph dari pada manifold tersebut. Disini diperoleh suatu proposisi yang penting yang memperluas sebuah hasil yang terkenal dari Lichnerowicz* [6,7].

A B S T R A C T

It is shown that the Schouten bracket of holomorphic tensors of a compact kählerian manifold W *defines a structure of graded complex Lie algebra on the space of holomorphic tensors of the manifold. We obtain an important proposition which generalizes a wellknown result of Lichnerowicz* [6,7].

Introduction

Given a compact kählerian manifold $(W,\underline{g})$ of complex dimension $\dim_C W = n$, Lichnerowicz [6,7] has obtained important

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properties of holomorphic tensors of W and its holomorphic forms as well. Under certain hypotheses given to the first Chern class $C_1(W)$ of W, some of the results generalize those obtained by Kodaira, Kobayashi and Calabi.

The fact that the Schouten bracket of holomorphic tensors of W defines a structure of graded complex Lie algebra on the space of such tensors leads us to investigate more characters of holomorphic tensors and forms of W. We obtain an important result, that is the proposition in §4a which generalizes a wellknown result of Lichnerowicz, and a fundamnetal theorem in §4d.

1. *Notion of complex manifold*

Suppose W is a compact and connected analytic complex manifold and let its complex dimension, $\dim_C W = n$. A domain U of W is a connected open set of W. We denote by C^n the space of n-tuples of complex numbers. In what follows, on indices we put the following convention: the greeks $\underline{\alpha}$, β, etc. = 1, ..., n, the latins a, b, etc. = 1, ..., 2n and $\bar{\alpha} = \alpha + n$.

a. A complex chart (or system of local complex coordinates) is defined on a domain U of W by:

$$\Psi_U : z \varepsilon U \rightarrow \{z^\alpha\} \ \varepsilon \ C^n$$

We write $\bar{z}^\alpha = z^{\bar{\alpha}}$. If U and V are respectively domains of two complex charts $\{z^\alpha\}$ and $\{z^{\beta'}\}$ with non - empty intersection, then the complex coordinates $\{z^\alpha\}$ of $z \varepsilon U \cap V$ of the first chart are holomorphic functions, wiht non-vanishing jacobian J^U_V, of the complex coordinates $\{z^{\beta'}\}$ of the same point z on the second chart. We write:

$$J^U_V = \det\ (\partial_{\beta'} z^\alpha)$$

where $\partial_\alpha = \partial/\partial z^\alpha$, $\partial_{\beta'} = \partial/\partial z^{\beta'}$. The complex structure of W furnishes the manifold itself with a natural orientation. On each point z of W, the complex structure of W determines a complex structure of its tangent space T_z. If T^c_z is the complexification of T_z, the complex structure of T_z is defined by the operator $J_z(J_z^2 = -Id)$ on the elements of T_z^c. The tensor field J of The operators J_z determines the "almost complex structure" of W. If S_z^c and $\bar{S}_z^c$ are respectively the proper subspaces of T_z^c corresponding to the proper values i and -i by J_z respectively, then we have:

$$T_z^c = S_z^c \oplus \bar{S}_z^c$$

This decomposition leads to the notion of type for the complex tensors and the operators on W.

b. A q-form of W is a complex exterior differential form of order q. A form of type (r,s) has the components with r indices in α and s indices in $\bar{\beta}$. If d is the operator of exterior differentiation, we then have $d = d' + d''$, where d' is of type (1,0) and d'' of type (0,1). From $d^2 = 0$, we deduce by considering the types, that $d'^2 = 0$, $d''^2 = 0$ and $d'd'' + d''d' = 0$.

A holomorphic r-form β is an r-form of type (r,0) such that $d''\beta = 0$. It is equivalently to say that: it is a form of type (r,0) such that in any complex chart (or simply, locally) admits local holomorphic functions as its components.

By abuse of terminology, we call an r-tensor A an antisymmetric contravariant r-tensor of W. *A holomorphic* r-tensor is an r-tensor of type (r,0) admitting, on a complex chart, components which are local holomorphic functions.

c. *A holomorphic transformation* of W is a transformation of W which leaves its complex structure invariant -or equivalently- leaves J invariant. *A holomorphic infinitesimal transformation* is defined by a real vector field X such that $L(X)J = 0$, where L(X) is the operator of Lie derivation with respect to X. This means that in a complex chart we have:

$$\partial_{\bar{\beta}} X^{\alpha} = 0 \qquad (1.1)$$

The relations (1.1) say that the part $X^{1,0}$ of type (1,0) of X is a holomorphic vector (1-tensor). Moreover JX is again a holomorphic infinitesimal transformation. Suppose L is the

Lie algebra of holomorphic infinitesimal transformations of W. If X, YεL, we then obtain the following identities of Lie brackets:

$$[JX,Y] = [X,JY] = J[X,Y]$$

and thus J defines on L a structure of complex Lie algebra. Let G be the largest connected group of holomorphic transformations of W. Bochner and Montgomery [1] have established that G admits a natural structure of complex Lie group, GxW $\to$ W being holomorphic. The algebra of G can be identified by the complex Lie algebra L (see also [4]).

d. We denote by H^r of complex dimension $b_{r,0}(W)$, the complex vector space of *closed* holomorphic r-forms of W. Let T^r be the space of holomorphic r-tensors of W. If $A \varepsilon T^r$ and $\beta \varepsilon H^r$, then $i(A)\beta$ (where $i(A)$ is the operator of exterior product by A) is a holomorphic scalar on W, and in fact since W is compact:

$$i(A)\beta = \text{const.}$$

We denote by I^r, the complex subspace of T^r defined by the elements A such that:

$$i(A)\beta = 0$$

for all elements β of H^r.

In particular to the elements $X^{1,0}$ of I, they correspond the elements X of the complex subspace I of L such that:

$$i(X)\beta = 0$$

for any *closed* holomorphic 1-form β. If, X, YεL and $\beta \varepsilon H^1$, we then have:

$$L(X)\beta(Y) - L(Y)\beta(X) - \beta([X,Y]) = 0$$

and hence:

$$i([X,Y])\beta = 0$$

Thus $[X,Y] \in I$. If $L' = [L,L]$ is the derived ideal of L, then $L' \subset I$ and I is an ideal of L such that L/I is abelian (see [6]). We see that if X is an element of L and admits a zero on W, then it necessarily belongs to I.

However, if $X \in L$ and $A \in T^r (r>1)$, $L(X)A$ does not necessarily belongs to I^r on a complex manifold; but later we see that, in the kählerian case, indeed it does, that is $L(X)A \in I^r$ (see [6]).

2. *Structure of graded Lie algebra of tensors* [2,3]

Let V be a differentiable manifold of dimension m. In what follows, we shall mean by a tensor, an antisymmetric contravariant tensor of V.

a. Suppose A and B are respectively r- and s-tensors of V. the *Schouten bracket* [9] of A and B, [A,B], is an (r+s-1)-tensor such that for any closed (r+s-1)-form μ of V, we have:

$$i([A,B])\mu = (-1)^{rs+s} i(A)di(B)\mu + (-1)^r i(B)di(A)\mu \tag{2.1}$$

The relation (2.1) determines uniquely the tensor [A,B]. One can easily find that on a domain U of a system of local coordinates $\{x^k\}$, [A,B] has the components:

$$[A,B]^{k_2\cdots k_{r+s}} = \frac{1}{(r-1)!s!}\,\varepsilon^{k_2\cdots k_{r+s}}_{i_2\cdots i_r j_1\cdots j_s}\,A^{ai_2\cdots i_r}\partial_a B^{j_1\cdots j_s}$$

$$+\frac{(-1)^r}{r!(s-1)!}\,\varepsilon^{k_2\cdots k_{r+s}}_{i_1\cdots i_r j_2\cdots j_s}\,B^{aj_2\cdots j_s}\partial_a A^{i_1\cdots i_r} \tag{2.2}$$

where ε is the indicator tensor of Kronecker. We deduce also:

$$[B,A] = (-1)^{rs}\,[A,B] \tag{2.3}$$

If C is a t-tensor, we then have the following formula:

$$(-1)^{st}[[C,A],B] + (-1)^{rs}[[B,C],A] + (-1)^{tr}[[A,B],C] = 0 \qquad (2.4)$$

Thus the space of tensors of V admits a *structure of graded Lie algebra* determined by the Schouten bracket. This bracket has been studied by Nijenhuis [8].

b. On the algebra of tensors of V, we define an operator L(A) on forms of V, where A is a tensor as follows: if A is an s-tensor and β is an r-form ($r \geq s-1$), then:

$$L(A)\beta = di(A)\beta - (-1)^{s}\, i(A)d\beta \qquad (2.5)$$

It is clear that $L(A)\beta$ is an (r+s-1)-form. For s=1, this operator reduces to the usual operator of Lie derivation with respect to a vector. On a domain U of a system of local coordinates $\{x^k\}$, $L(A)\beta$ has the components:

$$[L(A)\beta]_{k_s\ldots k_r} =$$

$$\frac{1}{(r-s)!s!}\,\varepsilon^{j_s\ldots j_r}_{k_s\ldots k_r}\,\partial_{j_s}\,A^{i_1\ldots i_s}\,\beta_{i_1\ldots i_s j_{s+1}\ldots j_r} +$$

$$-\frac{(-1)^{s}}{(s-1)!}\,A^{i_1\ldots i_s}\,\partial_{i_1}\beta_{i_2\ldots i_s k_s\ldots k_r} \qquad (2.6)$$

If $L(A)\beta = 0$, we then simply say that the form β is *invariant* by A.

Suppose K is an m-form on V of kernel k. Obviously:

$$L(A)K = di(A)K$$

On a domain U of a system of local coordinates $\{x^k\}$, we have:

$$i(A)K|_U = \frac{1}{s!}\,A^{i_1\ldots i_s}\,k\varepsilon_{i_1\ldots i_s i_{s+1}\ldots i_m}\,dx^{i_{s+1}}\wedge\ldots\wedge dx^{i_m}$$

Furthermore:

$L(A)K|_U =$

$$\frac{(-1)^{s-1}}{(s-1)!}\,\partial_t(kA^{ti_1\ldots i_{s-1}})\,\varepsilon_{i_1\ldots i_s i_{s+1}\ldots i_m}\,dx^{i_3}\wedge\ldots\wedge dx^{i_m}$$

Thus the m-form K is invariant by A if and only if on each domain U of a system of local coordinates $\{x^k\}$, we have:

$$\partial_t(kA^{ti_1\ldots i_{s-1}}) = 0 \qquad (2.7)$$

3. *The Kählerian case* [6,7]

a. Let W be a compact connected analytic complex manifold and $\dim_C W = n$. Consider the covariant real 2-tensors t of type (1,1). We introduce on the space of such tensors a real operator $\underline{J}$ (with $\underline{J}^2 = -Id$) defined by:

$$(\underline{J}t)(u,v) = t\,(Ju,v)$$

for any pair (u,v) of vectors u,v. If t is symmetric, then $\underline{J}t$ is antisymmetric and conversely.

On W, there exist hermitian metrics, i.e.: the riemannian metrics, of which the metric tensor $\underline{g}$ is of type (1,1). To the tensor $\underline{g}$, it corresponds by $\underline{J}$ a real 2-form $F = \underline{J}\underline{g}$ of type (1,1). In a complex chart $\{z^\alpha\}$ with domain U, we have:

$$\underline{g}|_U = 2g_{\alpha\bar\beta}\, dz^\alpha \otimes dz^{\bar\beta} \qquad F|_U = ig_{\alpha\bar\beta}\, dz^\alpha \wedge dz^{\bar\beta}$$

The manifold W is said to admit a *kählerian structure* defined by the metric $\underline{g}$ if the corresponding real 2-form $F(=\underline{J}\underline{g})$ is closed ($dF = 0$). The pair $(W,\underline{g})$ is called a *kählerian manifold* and the real 2-form F the fundamental form of W.

b. Suppose $(W,\underline{g})$ is a compact kählerian manifold, $\dim_C W = n$ with the fundamental form F. This form admits a rezo covariant derivative in the riemannian connection (we call: the kählerian connection) defined by the metric $\underline{g}$. Locally, the only coefficients which do not necessarily vanish of this connection, are those of pure type:

$$\Gamma^{\alpha}_{\beta\gamma} = g^{\alpha\bar{\rho}}\, \partial_{\beta} g_{\gamma\bar{\rho}} \text{ and } \Gamma^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}} = \bar{\Gamma}^{\alpha}_{\beta\gamma}$$

Let α and β be two r-forms of $(W,\underline{g})$. We denote by (α,β) the interior product of α and β (in general of two tensors; we consider only on forms). We review the following hermitian scalar product defined by:

$$<\alpha,\beta> = \int_W (\alpha,\beta)\eta \tag{3.1}$$

where η is the volume element of W.

If δ is the operator of codifferentiation on forms, we then have $\delta = \delta' + \delta''$, where δ' is of type (-1,0) and δ'' of type (0,-1). The operators δ, δ' and δ'' are respectively the transposes of d, d' and d'' with respect to (3.1). For a kählerian manifold $(W,\underline{g})$ the laplacian $\Delta = d\delta + \delta d$ of Hodge - de Rham on forms can be written as:

$$\Delta = 2(d'\delta' + \delta'd') = 2(d'' + \delta'') \tag{3.2}$$

and hence it is of type (0,0). From (3,2), in the kählerian case, it follows that any holomorphic form is harmonic and in particular it is closed. We obtain also that the part of type (1,0) of a real harmonic 1-form is holomorphic; the first Betti number of W, $b_1(W) = 2b_{1,0}(W)$, where $b_{1,0}(W) = p$ is known as the *irregularity* of the manifold.

If $X \in L$ and $\beta \in H^r$, then:

$$L(X)\beta = (di(X) + i(X)d)\beta = 0$$

Thus on a compact kählerian manifold, any holomorphic form is invariant by the algebra L (or by the group G).

c. Let A be an r-tensor of type (r,0). By duality defined by the metric and the complex conjugation it permits us to introduce an r-form $\sigma(A)$ of type (r,0) with:

$$\overline{\sigma(A)}_{\bar{\rho}_1\ldots\bar{\rho}_r} = g_{\bar{\rho}_1\sigma_1}\ldots g_{\bar{\rho}_r\sigma_r} A^{\sigma_1\ldots\sigma_r} \tag{3.3}$$

The map σ is an antilinear bijection from the space of r-tensors of type (r,0) onto the space of r-forms of type (r,0). For that A is to be holomorphic ($A\in T^r$), it is necessary and sufficient that locally:

$$\partial_{\bar{\beta}} A^{\rho_1\cdots\rho_r} = \nabla_{\bar{\beta}} A^{\rho_1\cdots\rho_r} = 0$$

where ∇ is the operator of covariant differentiation in the kählerian connection.

One example in utilizing σ, we see that the tensor A belongs to T^r if and only if the part of type (r+1,0) of $\sigma(A)$ is zero, that is to say:

$$[\nabla\sigma(A)]_{r+1,0} = 0 \tag{3.4}$$

We deduce that from the antisymmetrization, $\sigma(A)$ is necessarily d'-*closed* ($d'\sigma(A) = 0$). From the decomposition of G. de Rham, it follows that:

$$\sigma(A) = d'\mu + H\sigma(A) \qquad (A\in T^r) \tag{3.5}$$

where $H\sigma(A)$ is a holomorphic r-form. The followings holomorphic scalar

$$m(A) = i(A)H\sigma(A) = (H\sigma(A),H\sigma(A))$$

on a *compact* kählerian manifold is a constant. From (3.5), we obtain:

$$<H\sigma(A),H\sigma(A)> = <H\sigma(A),\sigma(A)> = Vm(A) \tag{3.6}$$

where V is the volume of W. Hence $m(A) = 0$ tells that $\sigma(A)$ is d'-homologous to zero. From, (3.6), it follows that *a holomorphic r-tensor* A *belongs to* I^r *if and only if* $m(A) = 0$ *or* $\sigma(A)$ *is* d'*-homologous to zero* [7].

The image of T^r under the map $A\in T^r \to H\sigma(A)\in H^r$ is a subspace Q^r of H^r. If $H\sigma(A)\neq 0$ is an element of Q^r, then according to (3.6), $m(A) = i(A)H\sigma(A)\neq 0$. *Thus a non-trivial member of* Q^r *does not have a zero on* W.

4. Holomorphic tensors leaving a real 2n-form $K \geq 0$ *invariant.*

Let $(W,\underline{g})$ be a compact kählerian manifold, $\dim_C W = n$. By linearity the Schouten bracket can be extended to complex tensors of the complex manifold W. If A is an r-tensor, we may also extend the operator L(A) to complex forms of W.

a. If $A \in T^r$ and $B \in T^s$, then on a domain U of a complex chart $\{z^\alpha\}$, the Schouten bracket [A,B] has the components (see (2.2)):

$$[A,B]^{\tau_2 \ldots \tau_{r+s}} = \frac{1}{(r-1)!s!} \varepsilon^{\tau_2 \ldots \tau_{r+s}}_{\rho_2 \ldots \rho_r \sigma_1 \ldots \sigma_s} A^{\lambda \rho_2 \ldots \rho_r} \partial_\lambda B^{\sigma_1 \ldots \sigma_s}$$

$$+ \frac{(-1)^r}{r!(s-1)!} \varepsilon^{\tau_2 \ldots \tau_{r+s}}_{\rho_1 \ldots \rho_r \sigma_2 \ldots \sigma_s} B^{\lambda \sigma_2 \ldots \sigma_s} \partial_\lambda A^{\rho_1 \ldots \rho_r} \qquad (4.1)$$

From (4.1), it follows that the components of [A,B] are local holomorphic functions and hence [A,B] is contained in T^{r+s-1}. Thus the Schouten bracket defines on the space of holomorphic tensors, *a structure of graded complex Lie algebra.* Moreover, the compact manifold W being kählerian, if β is a holomorphic (r+s-1)-form, then it is closed and from (2.1), it follows that:

$$i([A,B])\beta = (-1)^{rs+s} i(A)di(B)\beta + (-1)^r i(B)di(A)\beta$$

where the holomorphic forms $i(A)\beta$ and $i(B)\beta$ are closed. So that:

$$i([A,B]) = 0$$

We thus obtain the following important proposition generalizing a welknown result of Lichnerowicz [7]:

Proposition - On a compact kählerian manifold, if $A \in T^r$ *and* $B \in T^s$, *then the holomorphic* (r+s-1)*-tensor* [A,B] *is contained in* I^{r+s-1}. *In particular if* $X \in L$ *and* $A \in T^r$, *then the* r*-tensor* L(X)A *belongs to* I.

b. If $K \not\equiv 0$ is a real 2n-form ≥ 0 on W, then $K = f\eta$ for a scalar $f \geq 0$. Suppose A is a real r-tensor of W. From (2.7), it follows that K is invariant by A if and only if on a domain

U of a complex chart, we have:

$$\nabla_a (fA^{ai_2 \ldots i_r}) = 0 \tag{4.2}$$

Let L_f be the complex subalgebra of L leaving the form K = $f\eta$ invariant. If $X \varepsilon L_f$, then on U:

$$\nabla_a (fX^a) = \nabla_\alpha (fX^\alpha) + \nabla_{\bar\beta} (fX^{\bar\beta}) = 0$$

But L_f is a complex subalgebra, $JX \varepsilon L_f$ and hence:

$$\nabla_a (f(JX)^a) = i\nabla_\alpha (fX^\alpha) - i\nabla_{\bar\beta} (fX^{\bar\beta}) = 0$$

And clearly, we deduce for any $X \varepsilon L_f$, that:

$$\nabla_\alpha (fX^\alpha) = 0$$

that is equivalently under the intrinsic form, we obtain:

$$\delta' \{f\sigma(X^{1,0})\} = 0 \tag{4.3}$$

where $X^{1,0}$ is the part of type (1,0) of X.

More generally, suppose K is invariant by a real r-tensor (r>1)

$$A = A^{r,0} + A^{0,r}$$

the sum of its part of type (r,0), $A^{r,0}$, and its complex conjugate $A^{0,r} = \bar{A}^{r,0}$. We deduce from (4.2) that K is invariant by A if and only if on each domain U of a complex chart $\{z^\alpha\}$:

$$\nabla_\alpha (fA^{\alpha\rho_2 \ldots \rho_r}) = 0 \tag{4.4}$$

or intrinsically:

$$\delta' \{f\alpha(A^{r,0})\} = 0$$

c. Guided by the analyses in §4b, we introduce the complex subalgebra $U^r(f)$ $(f \geq 0)$ of T^r defined by the holomorphic r-tensors A satisfying:

$$\delta'\{f\sigma(A)\} = 0 \qquad (4.5)$$

for which we have given the interpretation. In what follows we denote by L_f the complex subalgebra of L defined by the holomorphic vectors satisfying (4.3).

Let $A \in U^r(f)$. If $A \in I^r$, then $\sigma(A)$ is d'-homologous to zero (see §3c). From (4.5), it follows that:

$$\delta'(fd'\mu) = 0$$

for a form μ. We obtain:

$$\langle fd'\mu, d'\mu \rangle = \langle \delta'(fd'\mu), \mu \rangle = 0$$

If U is an open set of W on which $f \neq 0$, then $d'\mu$ and hence A are zero on U. By analyticity, A vanishes on W and thus $U^r(f) \cap I^r = 0$. We establish [5]:

Lemma - If given a non-trivial scalar $f \geq 0$ *on a compact kählerian manifold* $(W, \underline{g})$, *we then ahve* $U^r(f)\ I^r = 0$. Moreover:

$$\dim_C U^r(f) \leq b_{r,0}(W)$$

A non-trivial element of $U^r(f)$ *never vanishes on* W.

In particular, the complex subalgebra L_f *of* L *which leaves the 2n-form* $K = f\eta \geq 0$ *invariant is such that* $L_f \cap I = 0$. L_f *is abelian and*

$$\dim_C L_f \leq b_{1,0}(W) = p$$

To see the inequality of dimensions of the lemma, we observe the antilinear map $A \in U^r(f) \rightarrow H\sigma(A) \in Q^r \subset H^r$. This map is injective since it has $U^r(f) \cap I^r$ as its kernel. The proof of the lemma is complete.

d. Suppose $f \not\equiv 0$ is a scalar ≥ 0 on $(W, \underline{g})$. If $A \in U^r(f)$ and

$B \varepsilon U^s(f)$, then locally we have:

$$\nabla_\lambda (fA^{\lambda\rho_2\cdots\rho_r}) = 0 \qquad \nabla_\lambda (fB^{\lambda\sigma_2\cdots\sigma_s}) = 0 \tag{4.6}$$

Introducting the kählerian connection of the manifold, (4.1) can be expressed as:

$$[A,B]^{\tau_2\cdots\tau_{r+s}} = \frac{1}{(r-1)!s!}\,\varepsilon^{\tau_2\cdots\tau_{r+s}}_{\rho_2\cdots\rho_r\sigma_1\cdots\sigma_s}\,A^{\lambda\rho_2\cdots\rho_r}\nabla_\lambda B^{\sigma_1\cdots\sigma_s}$$

$$+\frac{(-1)^r}{r!(s-1)!}\,\varepsilon^{\tau_2\cdots\tau_{r+s}}_{\rho_1\cdots\rho_r\sigma_2\cdots\sigma_s}\,B^{\lambda\sigma_2\cdots\sigma_s}\nabla_\lambda A^{\rho_1\cdots\rho_r} \tag{4.7}$$

From (4.6) and (4.7), a computation shows that:

$$f[A,B]^{\tau_2\cdots\tau_{r+s}} = \nabla_\lambda (fA \wedge B)^{\lambda\sigma_2\cdots\sigma_{r+s}}$$

Thus we derive the formula:

$$f\sigma([A,B]) = -\,\delta'\sigma(f(A \wedge B)) \tag{4.8}$$

Since $\delta'^2 = 0$, we deduce that:

$$\delta'\{f\sigma([A,B])\} = 0$$

and that $[A,B]\varepsilon\, U^{r+s-1}(f)$. According to the proposition of §4a, we see that $[A,B]\varepsilon\, I^{r+s-1}$. Hence by the lemma of §4c, we have $[A,B] = 0$. Consequently from (4.8), it follows that $\delta'\{f\sigma(A \wedge B)\} = 0$, which implies $A \wedge B\varepsilon U^{r+s}(f)$.

In particular if $X\varepsilon L_f$ and $A\varepsilon U(f)$, where:

$$U(f) = \bigoplus_{r=0}^{n} U^r(f)$$

then $L(X)A = [X,A] = 0$.

We have proved the following fundamnetal theorem:
Theorem - If f *is a non-negative scalar on a compact kählerian manifold* (W,g), $A \in U^r(f)$ *and* $B \in U^s(f)$, *then we have* $[A,B] = 0$ *and* $A \wedge B \in U^{r+s}(f)$. *Thus the exterior product of antisymmetric tensors of* W *defines on* U(f) *a structure of exterior algebra. In particular, the Lie algebra* L_f *leaves invariant each element of* U(f).

Acknowledgements

The author wishes to express his sincerest thanks to Professor A. Lichnerowicz of Collège de France, who has introduced him this fascinating problem in differential geometry while he was at Collège de France, Paris. The Financial support from the Institute of Technology Bandung is gratefully acknowledged.

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(Received 11th June 1974)